

Complex Plane: Roots of Complex Numbers

Roots of Unity and n -th Roots of Complex Numbers

Lecture Notes

September 15, 2026

1 Powers and Roots (continued)

Roots of Unity

Solving $z^n = 1$: $n \operatorname{Arg} z \in \{2\pi k, k \in \mathbb{Z}\}$. The solutions are the **n -th roots of unity** $1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}$, where $\omega_n := \operatorname{cis}(2\pi/n)$ is the principal root of unity.

Geometrically, these n points form a **regular n -gon inscribed in the unit circle**: multiplication by ω_n rotates the plane by $2\pi/n$, and it sends the set of roots to itself, since $\omega_n \cdot \omega_n^k = \omega_n^{k+1}$ just permutes $1, \omega_n, \dots, \omega_n^{n-1}$ cyclically. A finite set of points on a circle that is invariant under rotation by $2\pi/n$ – and evenly spans a full turn under n such rotations – is exactly the vertex set of a regular n -gon.

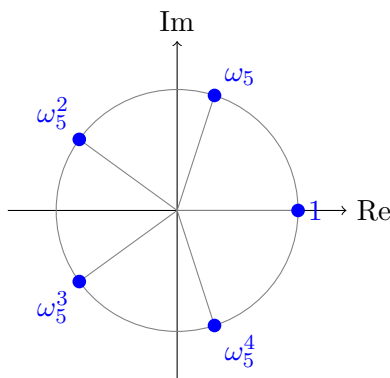


Figure 1: Example ($n = 5$): the 5th roots of unity $1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4$, equally spaced at angles $2\pi k/5$ on the unit circle.

Interactive illustration. To see how these roots are distributed in the complex plane: <https://www.geogebra.org/m/sy9qYMPe>.

n -th Roots of Complex Numbers

To find the n -th roots of $w \neq 0$ (i.e. $z^n = w$): if z_0, z_1 are two such roots, then $(z_1/z_0)^n = w/w = 1$, so z_1/z_0 is a root of unity. Hence the full set of n roots is generated from a single root z_0 as $z_0, z_0\omega_n, \dots, z_0\omega_n^{n-1}$.

Given $w = |w|(\cos \varphi + i \sin \varphi)$, a base root is

$$z_0 = |w|^{1/n} (\cos(\varphi/n) + i \sin(\varphi/n)).$$

Example ($n = 6, w = 1 + i$). Here $|w| = \sqrt{2}$ and $\varphi = \pi/4$, so $z_0 = 2^{1/12} \operatorname{cis}(\pi/24)$, and the six roots $z_0, z_0\omega_6, \dots, z_0\omega_6^5$ are equally spaced by angle $\pi/3$ on the circle of radius $2^{1/12}$.

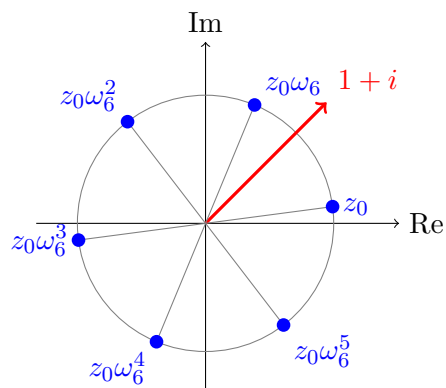


Figure 2: The six 6th roots of $1 + i$ (red), equally spaced on the circle of radius $2^{1/12}$.

Interactive illustration. To see how these roots are distributed in the complex plane: <https://www.geogebra.org/m/knWhMrsK>.